International Journal of Mathematics, Statistics and Operations Research
Vol. 1, No. 1, 2021 : pp. 49-60

ARF INDIA
Academic Open Access Publishing
www.arfjournals.com

SI_0^*–RINGS AND SI_0^*–MODULES

HAMZA HAKMI[1]

Article History

Received : 30 April 2021
Revised : 11 May 2021
Accepted : 18 May 2021
Published : 1 August 2021

ABSTRACT: This paper is a continuation of study of I_0^*–modules. We provide several characterization of module which it is endomorphism ring is a principal right (left) I_0^*–ring. New results obtained include necessary and sufficient conditions of module to be a principal right (left) I_0^*–module (I_0^*–module). Connectio between a principal right (left)module (module) and its endomorphism ring studied. Several basic properties of subsets $\Delta^*[M, N]$, $\nabla^*[M, N]$, $T_r^*[M, N]$, $J_r^*[M, N]$ of bi-module $[M, N]$ are proved, for every two modules M, N include when $T_r^*[M, N]$ is equal $J_r^*[M, N]$.

Keywords: I_0–Ring and Module, I_0–Ring and Module, (Co)retractable module, Semi-injective (projective) modules.

1. Introduction

The concept of I_0–rings was introduced by Nicholson in [8]. Recall that a ring R is an I_0–ring if, for every $a \in R$ there exists $b \in R$ such that $bab = b$. The concept of I_0–ring was extended by Hakmi to the module theory in [5]. Later, Zhou in [12] studied when the bi-module $[M, N]$ is an I_0 module. In [1] Abyzov introduced the notion of I_0^*–modules as dual to I_0 modules.

In this paper, we study the notion of I^*–rings and modules. In section 2, we prove that for a module M, E_M is a principal right I_0^*–ring if and only if for every non essential principal right ideal αE_M of E_M, $Im(\alpha)$ contained in a direct summand of $K \neq M$ of M. Also, for a module N, E_N is principal left I_0^*–ring if and only if for every non-essential principal left ideal $E_N\alpha$ of E_N, $Ker(\neq)$ contains a non-zero direct summand of N. In section 3, we study the concept of a principal right (left) I_0^*–bi-module $[M, N]$ as a natural generalization of the notion of right (left) (endomor-phism) ring. It is proved that $[M, N]$ is a principal right I_0^*–bi-module if and only if $T_r^*[M, N] = J_r^*[M, N]$. Thus, a ring R is a principal right I_0^*—ring if and only if $T_r^*(R) = J_r^*(R)$. Basic properties of $J_r^*[M, N]$ and $T_r^*[M, N]$ are proved in this section. In section 4, we prove that, if for a module N, $\nabla^*(E(N)) = J_r^*(E(N))$, then E_N is a principal right I_0^*–ring. It is also proved that, for a retractable module N, then E_N is a principal right I_0^*–ring if and only if N is a principal right I_0^*–

[1] Department of Mathematics, Damascus University, Damascus, Syria, E-mail: hhakmi-64@hotmail.com

To cite this article:

Hamza Hakmi. SI_0^*–Rings and SI_0^*–Modules. *International Journal of Mathematics, Statistics and Operations Research,* 2021. 1(1): 49-60

module and $\nabla^*(E_N) = J_r^*(E_N)$. Also if M is a co-retractable module, then E_M is a principal left I_0^*–ring if and only if M is a principal I_0^*–module and $\Delta^*(EM) = J_l^*(E_M)$. Basic properties of $\nabla^*[M, N]$ and $\Delta^*[M, N]$ are proved in this section.

Throughout this paper, R will be an associative R–ring with unityand modules are unitary right R–modules. For right R–modules M and N, we use the notation : $E_M = End_R(M)$ and $[M, N] = hom_R(M, N)$. Thus, $[M, N]$ is an $(E_N, E_M$-)bi-module. Let M_R be a module and N be a sub module of M. We say N is small in M if, whenever K is a sub module of M with $N + K = M$, then $K = M$. Dually, N is large in M if, $N \cap K$ implies $K = 0$, [6]. If M is an R–module, the radical of M, denoted $J(M)$, is defined to be the intersection of all maximal sub modules of N, [6]. We also denote for every sub module N of R–module M and $\ell(a) = \{x : x \in R;\ xa = 0\}$, $r(a) = \{x : x \in R;\ ax = 0\}$ for every $a \in R$.

2. Principal I_0^* Rings

Lemma 2.1. For any ring R the following are equivalent:

1. For every non-essential principal right ideal of aR of R, there is an idempotent $1 \neq e \in R$ such that $aR \subseteq eR$.
2. For every non-essential principal right ideal aR of R, there is an idempotent such that $0 \neq f \in R$.
3. For every non-essential principal right ideal aR of R, there is an idempotent $1 \neq g \in R$ such that $a = ga$.

Proof. Obvious.

We say that a ring R is a principal right I_0^*–ring, if it satisfies the conditions of Lemma 2.1. Similarly, we can define a principal left I_0^*–ring. A ring R is called a principal I_0^*–ring if it is a principal left and right I_0^*–ring. A ring R is (Von Neumann) regular, if for every $a \in R$, $a \in aRa$, it is clear that every regularring is a principal I_0^*–ring. Also, the ring of integers Z is a principal I_0^*–ring, because $\{0\}$ is a unique non-essential ideal in Z.

Lemma 2.2. For every module the following are equivalent:

1. E_M is a principal right I_0^*–ring.
2. For every non-essential principal right ideal $\alpha \in E_M$ of E_M, $I_m(\alpha)$ contained in a direct summand of $N \neq M$ of M.

Proof. (1) $\Rightarrow$ (2). Let $\alpha \in E_M$ with αE_M non-essential in E_M, then $\alpha = e\alpha$ for some idempotent $1 \neq e \in E_M$. So $Im(\alpha) \subseteq Im(e)$ and $Im(e) \neq M$ is a direct summand of M. (2) $\Rightarrow$ (1). Let $\alpha \in E_M$ be non-essential in E_M, then by assumption $Im(\alpha) \subseteq N$ for some direct summand of $N \neq M$ of M. Let $e : M \to N$ be the projection onto N. Since for every, $m \in M$, $\alpha(m) \in N = e(M)$, $\alpha = e\alpha$ by Lemma 2.1 our proof is completed.

Lemma 2.3. Let M_R be a module. Then the following are equivalent:

1. E_M is a principal left I_0^*–ring.
2. For every non-essential principal left ideal $E_M\alpha$ of E_M, $Ker(\alpha)$ contains a non-zero direct summand of M.

Proof. Is dual as in Lemma 2.2.

Recall that a module M_R is co-retractable [3], if for every proper sub module N of M, $\ell_{E_M}(N) \neq 0$. Also, recall that a module is retractable [9], if $hom_R(M, N) \neq 0$ for every sub module $N \neq 0$ of M.

Let M_R be a module and $E_M = End_R(M)$. Following Wisbauer [11], a module M is called semi-injective if for every $\alpha \in E_M$, $E_M\alpha = \ell_{E_M}(Ker(\alpha))$. Also, a module M is called semi-projective if for every $\alpha \in E_M$, $\alpha E_M = hom_R(M, Im(\alpha))$. In [5] it is proved that, if M is a semi-projective retractable module, then for every $\alpha \in E_M$, αE_M is large in E_M if and only if $I_M(\alpha)$ is large in M, [5, Lemma 5.1]. Also, if M is a semi-injective co-retractable module, then for every $\alpha \in E_M$, $E_M\alpha$ is large in E_M if and only if $Ker(\alpha)$ is small in M, [5, Lemma 5.7].

Using Lemma 2.2 and Lemma 2.3, we obtain the following results.

Corollary 2.4. Let M_R be a semi-projective retractable module. Then the following are equivalent:

1. E_M is a principal right I_0^*–ring.
2. For every non-essential principal right ideal αE_M of E_M, $Im(\alpha)$ contained in a direct summand $N \neq M$ of M.
3. For every $\alpha \in E_M$ with $Im(\alpha)$ non-essential in M, $Im(\alpha)$ contained in a direct summand $N \neq M$ of M.

Corollary 2.4. Let M_R be a semi-injective co-retractable module. Then the following are equivalent:

1. E_M is a principal left I_0^*–ring.
2. For every non-essential principal left ideal $E_M\alpha$ of E_M, $Ker(\alpha)$ contains a non-zero direct summand of M.
3. For every $\alpha \in E_M$ with $Ker(\alpha)$ not small in M, $Ker(\alpha)$ contains a non-zero direct summand of M.

3. Principal I_0^*–Modules

Recall that a module M_R is an I_0^*–module [1], if every non-essential sub module N of M contained in a direct summand $K \neq M$ of M. Here we say that a module M_R is a principal right I_0^* module if for every $\alpha \in E_M$ with

$Im(\alpha)$ is non-essential in M, $Im(\alpha)$ contained in a direct summand $K \neq M$ of M. Also, we say that a module M_R is a principal left I_0^*–module if for every $\alpha \in E_M$ with $Ker(\alpha)$ is not small in M, $Ker(\alpha)$ contains a non-zero direct summand of M. In [5] it is proved that, if M is a semi-injective co-retractable module, then $Ker(\alpha)$ is small in M if and only if $E_M\alpha$ is essential in E_M for every $\alpha \in E_M$, [5, Lemma 5.7]. Also, if M is a semi-projective retractable module, then $Im(\alpha)$ is essential in M if and only if αE_M is essential in E_M for every $\alpha \in E_M$, [5, Lemma 5.1].

From previous definitions and corollaries 2.4, 2.5 we derive the following:

Corollary 3.1. Let M_R be a semi-projective retractable module. Then the following statements are equivalent:

1. The module M is a principal right I_0^*–module.
2. The ring E_M is a principal right I_0^*–ring.

Corollary 3.2. Let M_R be a semi-injective co-retractable module.Then the following are equivalent:

1. The module M is a principal left I_0^*–module.
2. E_M is a principal left I_0^*–ring.

4. I_0^*–Bi-modules.

Letare M_R, N_R modules and $[M, N] = hom_R(M, N)$, then $[M, N]$ is an (E_N, E_M)-bi-module. Following [12], the Total of $[M, N]$ is a non-empty subset of $[M, N]$ given as follows:

$$\text{Tot}[M, N] = \{\alpha : \alpha \in [M, N];\ \alpha[N, M] \text{ contains no nonzer idempotents of } E_N\}$$
$$= \{\alpha : \alpha \in [M, N];\ [N, M]\alpha \text{ contains no nonzer idempotents of } E_M\}$$

Toward this subset we define:

$$T_r^*[M, N] = \{\alpha : \alpha \in [M, N];\ [N, M] \subseteq eE_N \text{ for some } 1 \neq e^2 = e \in E_N\}$$
$$T_l^*[M, N] = \{\alpha : \alpha \in [M, N];\ [N, M]\alpha \subseteq E_M e \text{ for some } 1 \neq e^2 = e \in E_N\}$$

It is clear that $T_r^*[M, N]$ and $T_l^*[M, N]$ are non-empty subsets in $[M, N]$.

Lemma 4.1. Let M_R, N_R, W_R are modules. Then:

1. $T_r^*[M, N]\cdot[W, M] \subseteq T_r^*[W, N]$.
2. $[N, W]\cdot T_l^*[M, N] \subseteq T_l^*[M, W]$.

Proof. (1) Let $\alpha \in T_r^*[M, N]$ and $\beta \in [W, M]$, then $\alpha\beta[N, W] \subseteq \alpha\,[N, M] \subseteq eE_N$ for some $1 \neq e^2 = \text{e} \in E_N$. Similarly, we can prove (2).

Let M_R, N_R are modules. Following [12], the Jacobson radical of $[M, N]$ is an ideal given as follows:

$$J[M, N] = \{\alpha : \alpha \in [M, N];\ \alpha\beta \in J(E_N) \text{ for all } \beta \in [N, M]\}$$

$= \{\alpha : \alpha \in [M, N]; \beta\alpha \in J(E_M)$ for all $\beta \in [N, M]\}$

Toward this ideal we define:

$J_r^*[M, N] = \{\alpha : \alpha \in [M, N]; \alpha[N, M]\}$ is non-essentialin $E_N\}$

$J_l^*[M, N] = \{\alpha : \alpha \in [M, N]; \alpha[N, M]\}\alpha$ is non-essentialin $E_M\}$

It is clear that if $N \neq 0$, then $J_r^*[M, N]$ is non-empty subset in $[M, N]$ and if $M \neq 0$, then $J_l^*[M, N]$ is non-empty subset in $[M, N]$.

Lemma 4.2. Let M_R, N_R, W_R are modules. Then the following hold:

1. $J_r^*[M, N]\cdot[W, M] \subseteq J_r^*[W, N]$.
2. $[N, W]\cdot J_l^*[M, N] \subseteq J_l^*[M, W]$.

Proof. (1) Let $\alpha \in J_r^*[M, N]$ and $\beta \in [W, M]$, then $\alpha\beta[N, W] \subseteq \alpha[N, M]$. Since $\alpha[N, M]$ is non-essential in E_N, $\alpha\beta[N, M]$ is non-essential in E_N. Similarly, we can prove (2).

Lemma 4.3. Let M_R, N_R are modules. Then the following hold:

1. $T_r^*[M, N] \subseteq J_r^*[M, N]$.
2. $T_l^*[M, N] \subseteq J_l^*[M, N]$.

Proof.

1. Let $\alpha \in T_r^*[M, N]$, then $\alpha[N, M] \subseteq eE_N$ for some $1 \neq e^2 = e \in E_N$, so $(1_N - e)E_N \neq 0$ and $\alpha[N, M] \cap [1_N - e)E_N = 0$. This shows that $\alpha[N, M]$ is non-essential in E_N, so $\alpha \in J_r^*[M, N]$. Similarly, we can prove (2).

Proposition 4.4. Let M_R, N_R are modules. Then the following are equivalent:

1. For every $\alpha \in [M, N]$ such that the right ideal $\alpha[N, M]$ is non-essential in E_N, there is an idempotent $1 \neq e \in E_N$ such that $\alpha[N, M] \subseteq eE_N$.
2. For every $\alpha \in [M, N]$ such that the right ideal $\alpha[N, M]$ is non-essential in E_N, there is an idempotent $0 \neq e \in E_N$ such that $e \in \ell_{E_N}(\alpha[N, M])$.
3. For every $\alpha \in [M, N]$ such that the right ideal $\alpha[N, M]$ is non-essential in E_N, there is a direct summand $K \neq N$ of N such that $Im(\alpha\beta) \subseteq K$ for all $\beta \in [N, M]$.
4. For every $\alpha \in [M, N]$ such that the right ideal $\alpha[N, M]$ is non-essential in E_N, there is a direct summand $K \neq N$ of such that $\Sigma_{\beta \in [N,M]} Im(\alpha\beta) \subseteq K$.
5. For every $\alpha \in [M, N]$ such that the right ideal $\alpha[N, M]$ is non-essential in E_N, there is an idempotent $1 \neq e \in E_N$ such that $Im(\alpha\beta) \subseteq K$ for all $\beta \in [N, M]$.

Proof. (1) $\Rightarrow$ (2). Let $\alpha \in [M, N]$ such that the right ideal is non-essential in E_N, then $\alpha[N, M) \subseteq eE_N$ for some idempotent $1 \neq e \in E_N$. So

$$1_N - e \in \ell_{E_N}(eE_N) \subseteq \ell_{E_N}(\alpha[N, M])$$

and $1_N - e \in E_N$ is a non-zero idempotent.

(2) $\Rightarrow$ (3). Let $\alpha \in [M, N]$ such that a right ideal $\alpha[N, M]$ is non-essential in E_N, then $e \in \ell_{E_N}(\alpha[N, M])$ for some idempotent $0 \neq e \in E_N$. So $Im(\alpha\beta) \subseteq Ker(e)$ for all $\beta \in [N, M]$ and $Ker(e) \neq N$ is a direct summand of N, hence $e \neq 0$.

(3) $\Rightarrow$ (4). Obvious. (4) $\Rightarrow$ (1). Let $\alpha \in [M, N)$ such that a right ideal is non-essential in E_N, then $\Sigma_{\beta \in [N,M]} Im(\alpha\beta) \subseteq K$ for some direct summand $K \neq N$ of N. Let $e : N \to K$ be the projection onto K, then $1_N - e \in E_N$ is an idempotent and $Im(\alpha\beta) \subseteq K = Im(e) = Ker(1_N - e)$, so $(1_N - e)\alpha\beta = 0$. This shows that $\alpha[N, M] \subseteq eE_N$. (2) $\Rightarrow$ (5) and (5) $\Rightarrow$ (2) are clear.

Let M_R, N_R are modules. We say that a bi-module $[M, N]$ is a principal right I_0^* bi-module, if it satisfies theconditions of Proposition 4.4. Similarly, we can define a principal left I_0^* bi-module of the form $[M, N]$.

Note that from previous definitions we derive the following:

Corollary 4.5. Let M_R be a module. Then a ring E_M is a principal right (left) I_0^*–ring if $[M, N]$ and only if is a principal right (left) I_0^*–bi-module.

Lemma 4.6. Let M_R, N_R are modules. If there is an epimorphism $\lambda \in [N, M]$, then the following are equivalent:

1. $[M, N]$ is a principal right I_0^*–bi-module.
2. For every $\alpha \in [M, N]$ such that the right ideal $\alpha[N, M]$ is non-essential in E_N, there is an idempotent $1 \neq e \in E_N$ such that $\alpha = e\alpha$.
3. For every $\alpha \in [M, N]$ such that the right ideal $\alpha[N, M]$ is non-essential in E_N, there is a direct summand $K \neq N$ of N such that $Im(\alpha) \subseteq K$.

Proof. (1) $\Rightarrow$ (2). Let $\alpha \in [M, N]$ such that the right ideal $\alpha[N, M]$ is non-essential in E_N, then by Proposition 4.4 $\alpha\beta = e\alpha\beta$ for some idempotent $1 \neq e \in E_N$ and for all $\beta \in [N, M]$. Since $\lambda \in [N, M]$, $\alpha\lambda = e\alpha\lambda$. Let $m \in M$, since λ is an epimorphism $\lambda(x) = m$ for some $x \in N$, so $\alpha(m) = \alpha\lambda(x) = e\alpha\lambda(x) = e\alpha(m)$, and so $\alpha = e\alpha$.

(2) $\Rightarrow$ (3). It is clear. (3) $\Rightarrow$ (1). Let $\alpha \in [M, N]$ such that the right ideal $\alpha[N, M]$ is non-essential in E_N, then by assumption $Im(\alpha) \subseteq K$ for some direct summand $K \neq N$ of N, so for all $\beta \in [N, M]$, $Im(\alpha) \subseteq K$ by Proposition 4.4 our proof completed.

Note that from Lemma 4.6 we derive the following:

Corollary 4.7. Let N_R be a module. Then the following are equivalent:

1. The ring E_N is a principal right I_0^*–ring.
2. For every $\alpha \in E_N$ such that the right ideal αE_N is non-essential in E_N, $\alpha = e\alpha$ for some idempotent $1 \neq e \in E_N$.
3. For every $\alpha \in E_N$ such that the right ideal $\alpha \in E_N$ is non-essential in E_N,

$Im(\alpha) \subseteq K$ for some direct summand $K \neq N$ of N.

Lemma 4.8. Let M_R, N_R are modules. If there is a monomorphism $\lambda \in [N, M]$, then the following are equivalent:

1. $[M, N]$ is a principal left I_0^*–bi-module.
2. For every $\alpha \in [M, N]$ such that the left ideal $[N, M]\alpha$ is non-essential in E_M there is an idempotent $1 \neq e \in E_M$ such that $\alpha = \alpha e$.
3. For every $\alpha \in [M, N]$ such that the left ideal $[N, M]\alpha$ is non-essential in E_M there is a non-zero direct summand K of N such that $K \subseteq Ker(\alpha)$.

Proof. Is dual as in Lemma 4.6.

Note that from Lemma 4.8 we derive the following:

Corollary 4.9. Let M_R be a module. Then the following are equivalent:

1. The ring E_M is a principal left I_0^*–ring.
2. For every $\alpha \in E_M$ such that the left ideal $E_M\alpha$ is non-essential in E_M, $\alpha = \alpha e$ for some idempotent $1 \neq e \in E_M$.
3. For every $\alpha \in E_M$ such that the left ideal $E_M\alpha$ is non-essential in E_M, $Ker(\alpha)$ contains a non-zero direct summand of M.

Lemma 4.10. Let M_R, N_R are modules. Then the following hold:

1. If there is an epimorphism $\lambda \in [N, M]$ and E_N is a principal right I_0^*–ring, then $[M, N]$ is a principal right I_0^*–bi-module.
2. If there is a monomorphism $\lambda \in [N, M]$ and E_M is a principal left I_0^*–ring, then $[M, N]$ is a principal left I_0^*–bi-module.

Proof. (1) Let $\alpha \in [M, N]$ such that $\alpha[N, M]$ is non-essential in E_N. Since $\alpha\lambda \in E_N$, $\alpha\lambda E_N$ is non-essential in E_N by assumption $\alpha\lambda E_N \subseteq eE_N$ for some idempotent $1 \neq e \in E_N$, so $\alpha\lambda = e\alpha\lambda$. In addition, since for every $m \in M$ there is $x \in N$ such that $\lambda(x) = m$, $\alpha(m) = \alpha\lambda(x) = e\alpha\lambda(x) = e\alpha(m)$, thus $\alpha = e\alpha$, so by Lemma 4.6 our proof completed. Similarly we can prove (2).

Theorem 4.11. Let M_R, N_R are modules. Then the following are equivalent:

1. $[M, N]$ is a principal right I_0^*–bi-module.
2. $T_r^*[M, N] = J_r^*[M, N]$.

In particular, E_M is a principal right I_0^*–ring if and only $T_r^*(E_M) = J_r^*(E_M)$.

Proof. (1) $\Rightarrow$ (2). $T_r^*[M, N] \subseteq J_r^*[M, N]$ by Lemma 4.3. Let $\alpha \in J_r^*[M, N]$, then $\alpha[N, M]$ is non-essential in E_N, by assumption $\alpha[N, M] \subseteq eE_N$ for some idempotent $1 \neq e \in E_N$, so $\alpha \in T_r^*[M, N]$.

(2) $\Rightarrow$ (1). Let $\alpha \in [M, N]$ such that $\alpha[N, M]$ is non-essential in, then $\alpha \in J_r^*[M, N] = T_r^*[M, N]$, so $\alpha[N, M] \subseteq gE_N$ for some idempotent $1 \neq g \in E_N$.

Theorem 4.12. Let M_R, N_R are modules. Then the following are equivalent:

1. $[M, N]$ is a principal left I_0^*–module.

2. $T_l^*[M, N] = J_l^*[M, N]$

In particular, E_M is a principal left I_0^*–ring if and only if $T_l^*(E_M) = J_l^*(E_M)$.

Proof. Is dual as in Theorem 4.11.

5. (Co-) Singular Subsets

Let M_R, N_R are modules. Following [12], the singular ideal of $[M, N]$ given as follows:

$$\Delta[M, N] = \{\alpha : \alpha \in [M, N];\ Ker(\alpha) \text{ is essential in } M]$$

The co-singular ideal of $[M, N]$ is

$$\nabla[M, N] = \{\alpha : \alpha \in [M, N];\ Ker(\alpha) \text{ is small in } N]$$

Toward this ideal we define:

$$\Delta^*[M, N] = \{\alpha : \alpha \in [M, N];\ Ker(\alpha) \text{ is non-small in } M]$$

$$\nabla^*[M, N] = \{\alpha : \alpha \in [M, N];\ Im(\alpha) \text{ is non-essential in } N]$$

It is clear that if $M \neq 0$, then $\Delta^*[M, N]$ is a non-empty subset in $[M, N]$ and if $N \neq 0$, then $\nabla^*[M, N]$ is a non-empty subset in $[M, N]$.

Lemma 5.1. Let M_R, N_R, W_R be modules. Then the following hold:

1. $[N, W]\cdot\Delta^*[M, N] \subseteq \Delta^*[M, W]$.
2. If there is a monomorphism $\lambda \in [N, M]$ then $T_l^*[M, N] \subseteq \Delta^*[M, N]$.
3. $T_l^*(E_M) \subseteq \Delta^*(E_M)$.

Proof. (1) Let $\beta \in [N, W]$ and $\alpha \in \Delta^*[M, N]$. Since $Ker(\alpha) \subseteq Ker(\alpha\beta)$ and $Ker(\alpha)$ is non-small in M, $Ker(\beta\alpha)$ is non-small in M.

(2) Let $\lambda \in [N, M]$ be is a monomorphism and $\alpha \in T_1^*[M, N]$, then $[N, M]\alpha \subseteq E_M e$ for some idempotent $1 \neq e \in E_M$. Since $\lambda\alpha = \lambda\alpha e$ and λ monomorphism, $\alpha = \alpha e$ and $Ker(e) \subseteq Ker(\alpha)$. Hence $e \neq 1$, $Ker(e)$ is non-small in M, so $Ker(\alpha)$ is non-small in M, thus $\alpha \in \Delta^*[M, N]$. (3) Follows from (2) for $M = N$.

Lemma 5.2. Let M_R, N_R, W_R be modules. Then the following hold:

1. $\nabla^*[M, N]\cdot[W, M] \subseteq \nabla^*[W, N]$.
2. If there is an epimorphism, $\lambda \in [N, M]$, then $T_r^*[M, N] \subseteq \nabla^*[M, N]$.
3. $T_r^*(E_M) \subseteq \nabla^*(E_M)$.

Proof. Is dual as in Lemma 5.1.

Following [2], recall that a module M_R is an I_0–module, if every non-small sub module of M contains a nonzero direct summand of M.

Proposition 5.3. Let M_R be an I_0–module. Then the following hold:

1. For every $N \in \text{mod} - R$, $\Delta^*[M, N] \subseteq T_l^*[M, N]$.

2. For every $N \in$ mod – R, $\Delta^*[M, N] \subseteq J_l^*[M, N]$.
3. $\Delta^*(E_M) = T_1^*(E_M)$.
4. If $\Delta^*(E_M) = J_l^*(E_M)$, then E_M is a principal left I_0–ring.

Proof. (1) Let $\alpha \in \Delta^*[M, N]$. Since Ker($\alpha$) is non-small in M and M is an I_0–module, $K \subseteq Ker(\alpha)$ for some direct summand $K \neq 0$ of M. Let $e : M \to K$ be the projection onto K, then $\alpha \neq 0$ and $\beta\alpha = \beta\alpha(1_M - e) \in E_M(1_M - e)$ for all. So $[N, M]\alpha \subseteq E_M (1_M - e)$ where $1_M - e \neq 1$ is an idempotent of E_M, therefore $\alpha \in T_1^*$[M, N]. (2) Follows from Lemma 4.3 and (1). (3) Follows from Lemma 5.1 and (1) for $M = N$. (4) Let $\alpha \in E_M$ such that $E_M\alpha$ is non-essential in E_M, then $\alpha \in J_l^*(E_M) = \Delta^*(E_M)$ and so $Ker(\alpha)$ is non-essential in M. Since M is an I_0–module, $e(M) \subseteq Ker(\alpha)$ where $0 \neq e^2 = e \in E_M$. Hence $\alpha e \neq 0$, $\alpha = \alpha(1_M - e)$ and $1 \neq 1_M - e \in E_M$ is idempotent. So $E_M\alpha \subseteq E_M(1_M - e)$ and our proof completed.

Following \cite{Ab}, recall that a module M_R is an module, if every non-essential sub module of M contained in a direct summand $K \neq M$ of M.

Proposition 5.4. Let N_R be an I_0^*–module. Then the following hold:

1. For every $M \in$ mod – R, $\Delta^*[M, N] \subseteq T_r^*[M, N]$.
2. For every $M \in$ mod – R, $\nabla^*[M, N] \subseteq J_r^*[M, N]$.
3. $\nabla^*[E_N] = T_r^*[E_N]$.
4. If $\nabla^*[E_N] = J_r^*[E_N]$, then E_N is a principal right I_0^*–ring.

Proof. Is dual as in Proposition 5.3.

Lemma 5.5. Let M_R, N_R be modules and $\alpha \in [M, N]$. Then:

1. If N is retractable and $\alpha[N, M]$ is essential in E_N, then $Im(\alpha)$ is essential in N.
2. If M is co-retractable and $[N, M]\alpha$ is essential in E_M, then Ker(α) is small in M.

Proof. (1) Let K be a sub module of N such that $Im(\alpha) \cap K = 0$. Suppose that $K \neq 0$, since N is retractable, $[N, K] = hom_R(N, K) \neq 0$. Let $\lambda \in \alpha[N, M] \cap [N, K]$, then $Im(\lambda) \subseteq Im(\alpha) \cap K = 0$, so λ. Hence $\alpha[N, M]$ is essential in E_N, $[N, K] = 0$ a contradiction, so $K \neq 0$. (2) Is dual as in (1).

Lemma 5.6.

(I) Let N_R be a retractable module.Then for every $M \in$ mod – R,

$$\nabla^*[M, N] \subseteq J_r^*[M, N]$$

In particular, $\nabla^*(E_N) = J_r^*(E_N)$.

(II) Let M_R be a co-retractable module. Then for every N mod –R,

$$\Delta^*[M, N] \subseteq J_r^*[M, N]$$

In particular, $\Delta^*(E_M) = J_l^*(E_M)$.

Proof. Obvious by Lemma 5.5.

Theorem 5.7. Let N_R be a retractable module. Then the following are equivalent:

1. N is a principal right I_0^*–module and $\nabla^*(E_N) = J_r^*(E_N)$.
2. E_N is a principal right I_0^*–ring.

Proof. (1) $\Rightarrow$ (2). Let $\alpha \in E_N$ such that αE_N is non-essential in E_N, then by assumption $Im(\alpha)$ is non-essential in N. So $Im(\alpha) \subseteq e(N)$ for some idempotent $1_N \neq e \in E_N$ and so $\alpha E_N = \subseteq eE_N$.

(2) $\Rightarrow$ (1). Since is retractable, by Lemma 5.6 we have $\nabla^*(E_N) \subseteq J_r^*(E_N)$. Let $\alpha \in J_r^*(E_N)$, since αE_N is non-essential in E_N and by assumption, $\alpha E_N \subseteq eE_N$ for some idempotent $1_N \neq e \in E_N$, so $Im(\alpha) \subseteq Im(e)$, where $Im(e) \subseteq N$ is a direct summand of N. Hence $Im(\alpha) \cap Im(1_N - e) = 0$, $Im(\alpha)$ is non-essential in N, so $\alpha \in \nabla^*(E_N)$. Let $\lambda \in E_N$ such that $Im(\lambda)$ is non-essential in N, then $\lambda \in J_r^*(E_N)$, so λE_N is non-essential in E_N by assumption $\lambda E_N \subseteq gE_N$ for some idempotent $1_N \neq e \in E_N$, therefore $Im(\lambda) \subseteq Im(g)$, where $Im(g) \neq N$ is a direct summand of N, thus N is a principal right I_0^*–module and our proof is completed.

Theorem 5.8. Let M_R be a co-retractable module. The following are equivalent:

1. M is a principal left I_0^*–module and.
2. E_M is a principal left I_0^*–ring.

Proof. Is dual as in Theorem 5.8.

Proposition 5.9. Let F_R be a free module. Then for every $M \in$ mod $-R$, $J_r^*[M, F] = \nabla^*[M, F]$. In particular $J_r^*(E_F) = \nabla^*(E_F)$.

Proof. Since any free module is retractable, so by Lemma 5.6 we only need to show that $J_r^*[M, F] \subseteq \nabla^*[M, F]$. Let $\alpha \in J_r^*[M, F]$, then $\alpha[F, M]$ is non-essential in E_F.

So there exists a right ideal $I \neq 0$ of E_F such that $\alpha[F, M] \cap I \neq 0$. Since $I \neq 0$, there is $\beta \in I$ such that $Im(\beta) \neq 0$ and $[F, Im(\beta)] = hom_R(F, Im(\beta)) \neq 0$ hence F is retractable. So there is $0 \neq \lambda \in [F, Im(\beta)]$ and $\lambda E_F \subseteq \beta E_F \subseteq I$, hence $Im(\lambda) \subseteq Im(\beta)$ and F is free, since $\alpha[F, M] \cap \lambda E_F \subseteq \alpha[F, M] \cap I = 0$, $\alpha[F, M] \cap \lambda E_F = 0$. On the other hand, since F is free and retractable,

$$\begin{aligned} hom_R(F, Im(\alpha) \cap Im(\lambda)) &= hom_R(F, Im(\alpha)) \cap hom_R(F, Im(\lambda)) \\ &= \alpha[F, M] \cap \lambda E_F = 0 \end{aligned}$$

therefore $Im(\alpha) \cap Im(\lambda) = 0$. This shows that $Im(\alpha)$ is non-essential in F, hence $Im(\lambda) \neq 0$, so $\alpha \in \nabla^*[M, F]$.

Proposition 5.10. Let Q_R be an injective co-retractable module. Then for every $N \in \text{mod} - R$, $J_l^*[Q, N] = \Delta^*[Q, N]$. In particular, $J_l^*(E_Q) = \Delta^*(E_Q)$.

Proof. Is dual as in Proposition 5.9.

Theorem 5.11. For any free module F_R the following are equivalent:

1. F is a principal right I_0^*–module.
2. E_F is a principal right I_0^*–ring.

Proof. (1) $\Rightarrow$ (2). Since any free module is retractable, by Proposition 5.9 we have

$$\Delta^*(E_F) = J_r^*(E_F)$$

and by Theorem 5.7 follows that E_F is a principal right I_0^*–ring. (2) $\Rightarrow$ (1). Obvious by Theorem 5.7.

It is easy to check that the Theorem 5.11 is true for any projective module $P \neq 0$ such that $J(P) = 0$.

Theorem 5.12. Let Q_R be an injective co-retractable module. Then the following are equivalent:

1. Q is a principal left I_0^*–module.
2. E_Q is a principal left I_0^*–ring.

Proof. Is dual as in Theorem 5.11.

Corollary 5.13. Let F_R be a free module. Then the following are equivalent:

1. $[M, F]$ is a principal right I_0^*–bi-module, for all $M \in \text{mod} -R$.
2. $J_r^*[M, F] = T_r^*[M, F]$ for all $M \in \text{mod} -R$.
3. $J_r^*[M, F] = \nabla^*[M, F]$ for all $M \in \text{mod} -R$.

Proof. (1) $\Rightarrow$ (2). Obvious by Theorem 4.12. (2) $\Rightarrow$ (3). Follows from Proposition 5.9. (3) $\Rightarrow$ (1). Let $\alpha \in [M, F]$ such that $\alpha[F, M]$ is non-essential in E_F, then by Lemma 5.5 $Im(\alpha)$ is non-essential in F, so by assumption $\alpha \in T_r^*[M, F]$ and $\alpha[F, M] \subseteq eE_F$ for some idempotent $1_F \neq e \in E_F$, this shows that $[M, F]$ is a principal right I_0^*–bi-module.

Corollary 5.14. Let Q_R be an injective co-retractable module. Then the following are equivalent:

1. $[Q, N]$ is a principal left I^*–bi-module, for all.
2. $J_l^*[Q, N] = T_l^*[Q, N]$, for all $N \in \text{mod} - R$.
3. $T_l^*[Q, N] = \Delta^*[Q, N]$, for all $N \in \text{mod} - R$.

Proof. Is dual as in Corollary 5.13.

Following [7], recall a module N_R is locally injective if, for every sub module $A \subseteq N$, which is non-essential in N, there exists an injective sub module $0 \neq Q \subseteq N$ with $A \cap Q = 0$. Also, recall a module M_R is locally projective if, for every sub module $B \subseteq M$, which is non small in M, there exists a projective direct summand $0 \neq P \subseteq M$ with $P \subseteq B$.

Proposition 5.15.

(I) Let M_R be a locally projective module. Then for every $N \in \text{mod} - R$, $\Delta^*[M, N] \subseteq T_l^*[M, N]$. In particular $\Delta^*(E_M) = T_l^*(E_M)$.

(II) Let N_R be a locally injective module. Then for every $M \in \text{mod} - R$, $\nabla^*[M, N] \subseteq T_r^*[M, N]$. In particular, $\nabla^*(E_N) = T_r^*(E_N)$.

Proof. (I) Let $\alpha \in \Delta^*[M, N]$, then $Ker(\alpha)$ is non small in M, so $P \subseteq Ker(\alpha)$ for some projective direct summand $P \neq 0$ of M. Suppose that $e : M \to P$ the projection onto P, then $\alpha e = 0$ and so $\alpha = \alpha(1_M - e)$. Since

$$[N, M]\alpha = [N, M]\alpha(1_M - e) \subseteq E_M(1_M - e)$$

where $1 \neq (1_M - e) \in E_M$ is an idempotent, $\alpha \in T_l^*[M, N]$. Thus for $M = N$, $\Delta^*(E_M) \subseteq T_l^*(E_M)$ and by Lemma 5.1 our proof is completed. Similarly we can prove (II).

References

[1] Abyzov, A. N. (2014), Modules, Russian Mathematics. (Iz. VUZ), Vol. 58, No. 8, p. 1-14.

[2] Abyzov, A. N. and Tuganbaev, A. A. (2008), Rings over Which all Modules are modules II, Fundam. Prikl. Mat.14, No. 2, 3 – 12.

[3] Amini, B. and Ershad, M.(2009), Co-retractable Modules, J. Aust. Math. Soc. 86, 289–304.

[4] Azumaya, G. (1991), Semi-Perfect Modules, *J. Algcbra*. 136, 73 – 58.

[5] Hamza, H. (1998), Rings and Modules, *Math. J.* Okayama Univ. Vol. 40, 91 – 97.

[6] Kasch, F. (1982), Modules and Rings, London Math. Soc. Mono.

[7] Kasch, F. (2002), Locally injective modules and locally projective modules, Rocky *Mountain J. Math.* 32, (4), 1493 – 1504.

[8] Nicholson, W. K. (1975), Rings, Trans. Amer. Math. Soc. 207, 361 – 373.

[9] Vedadi, M. R. (2007), Essentially Retractable Modules, Iran. Journal of Science.18, 355 – 360.

[10] Ware, R. (1971), Endomorphism Rings of Projective Modules, Trans. Amer. Math. Soc.155, 233 – 256.

[11] Wisbauer, R. (1991), Foundations of Modules and Rings Theory, Philadelphia: Gordon and Breach.

[12] Zhou, Y. (2009), On (Semi) Regularity and The Total of Rings and Modules, *Journal of Algcbra*. 322, 562 – 578.